A canonical system of basic invariants of a finite reflection group

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Abstract

A canonical system of basic invariants is a system of invariants satisfying certain differential equations. Some researchers studied properties of a canonical system related to the mean value property for polytopes. In this article, we naturally identify the vector space spanned by a canonical system of basic invariants with an invariant space determined by a fundamental alternating polynomial. From this identification, we give explicit formulas of canonical systems of basic invariants. The construction of the formulas does not depend on the classification of irreducible finite reflection groups.

Key Words: basic invariants, invariant theory, finite reflection groups.

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1 Introduction

Let V be a real n-dimensional Euclidean space, and $W \subseteq O(V)$ a finite reflection group. Let S denote the symmetric algebra $S(V^*)$ of the dual space V^*, and S_k the vector space consisting of homogeneous polynomials of degree k. Then W naturally acts on S and S_k. According to Chevalley [2], the subalgebra $R = S^W$ of W-invariant polynomials of S is generated by n algebraically independent homogeneous polynomials. A system of such generators is called a system of basic invariants of R. It is easy to construct systems of basic invariants for reflection groups of

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types A_n, B_n, D_n and I_2. Many researchers constructed an explicit system of basic invariants for a reflection group of each type (Coxeter [3], Mehta [12], Saito, Yano, and Sekiguchi [14], and Sekiguchi and Yano [15, 16])

Let $v_1, \dots, v_n$ be an orthonormal basis for V, $x_1, \dots, x_n$ the basis for V^* dual to $v_1, \dots, v_n$, and $\partial_1, \dots, \partial_n$ the basis for V^{**} dual to $x_1, \dots, x_n$. Since V is a inner product space, we can identify V^{**} with V naturally. To avoid a confusion, we distinguish the basis $\partial_1, \dots, \partial_n$ from the basis $v_1, \dots, v_n$. We define an inner product $\langle \cdot, \cdot \rangle : S \times S \to \mathbb{R}$ by

$$\langle f, g \rangle = f(\partial) g|_{x=0} \qquad (f, g \in S), \tag{1.1}$$

where $x = (x_1, \dots, x_n)$ and $\partial = (\partial_1, \dots, \partial_n)$. Define a bilinear map $(\cdot, \cdot) : S \times S \to S$ by

$$(f, g) = f(\partial) g \qquad (f, g \in S). \tag{1.2}$$

Two systems $g_1, \dots, g_n$ and $h_1, \dots, h_n$ of basic invariants are said to be equivalent if there exists $A \in \mathrm{GL}_n(\mathbb{R})$ such that

$$[h_1, \dots, h_n] = [g_1, \dots, g_n] A.$$

Definition 1.1. *A system $f_1, \dots, f_n$ of basic invariants is said to be canonical if it satisfies the following system of partial differential equations:*

$$(f_i, f_j) = \langle f_i, f_j \rangle \delta_{ij} \tag{1.3}$$

for $i, j = 1, \dots, n$ where δ_{ij} is the Kronecker symbol.

A canonical system was introduced by Flatto [4, 5] and Flatto and Wiener [6] for determining the structure of the linear space consisting of $P(0)$-harmonic functions, where $P(k)$ is a k-skeleton of an n-dimensional polytope P for a nonnegative integer $0 \leq k \leq n-1$. (An $\mathbb{R}$-valued continuous function is called a $P(k)$-harmonic function if it satisfies the mean value property on $P(k)$.) Flatto [4, 5] and Flatto and Wiener [6] verified that there exists a unique (up to equivalence) canonical system of basic invariants for any finite reflection group, and gave an algorithm. In [4, 5, 6], a canonical system of basic invariants was treated as a solution of a certain system of partial differential equations. The formulation above is due to Iwasaki [9].

Explicit formulas of canonical systems play an important part in the study for determining the structure of the linear space consisting of $P(k)$-harmonic functions. Especially, from the argument using explicit formulas of canonical systems, the

structure of the linear space consisting of $P(k)$-harmonic functions was determined for any $0 \leq k \leq n-1$ when P is a regular convex polytope (see [8, 10, 11]). (When P is a regular convex polytope, the symmetry group of P is a finite reflection group of type A, B, F, H, or I.)

However, the algorithm for constructing a canonical system (given by Flatto [4, 5] and Flatto and Wiener [6]) does not seem to be effective in practice. (It is hard to give an explicit formula of a canonical system from the algorithm.) It is an interesting problem to determine canonical systems. Iwasaki [9] gave explicit formulas of canonical systems for reflection groups of types A_n, B_n, D_n and I_2. Iwasaki, Kenma and Matsumoto [11] gave explicit formulas of canonical systems for the irreducible finite reflection groups of types F_4, H_3 and H_4. The problem for determining canonical systems of basic invariants for the remaining types (E_6, E_7 and E_8) has been open. In this article, we explicitly construct canonical systems of basic invariants from an arbitrary system of basic invariants (Theorem 3.5 and Theorem 3.11).

In the case of the types except D_n with even n, we can uniformly construct the canonical systems from an arbitrary system of basic invariants (Theorem 3.5). We need to consider the case of type D_n with even n separately, because the degrees of two invariants in a system meet at n. Iwasaki [9] constructed a canonical system of basic invariants containing the monomial $x_1 \cdots x_n$. By using the monomial $x_1 \cdots x_n$, we can also construct a canonical system from an arbitrary system of basic invariants (Theorem 3.11).

Let Φ be the root system associated with W, and Φ^+ a positive system. For $\alpha \in \Phi$, fix a homogeneous polynomial L_α of degree 1 defining the reflecting hyperplane H_α (i.e., $\ker L_\alpha = H_\alpha$). A polynomial $f \in S$ is said to be alternating if $wf = (\det w)f$ for all $w \in W$. Set $\Delta = \prod_{\alpha \in \Phi^+} L_\alpha$, the product of polynomials L_α, then Δ is an alternating polynomial. We call Δ the fundamental polynomial. Set

$$\mathbb{R}[\partial]\Delta := \{ f(\partial)\Delta \mid f \in S \}.$$

We naturally identify the vector space spanned by a canonical system of basic invariants with $(\mathbb{R}[\partial]\Delta \otimes_{\mathbb{R}} V^*)^W$ (Lemma 2.3). To determine canonical systems, we investigate $\mathbb{R}[\partial]\Delta$. It is known that the graded vector space $\mathbb{R}[\partial]\Delta$ affords the regular representation of W (see Bourbaki [1] and Steinberg [17]). This is one of the keys for determining canonical systems.

2 Characterization of the canonical systems

Let R_+ be the ideal of R generated by homogeneous elements of positive degrees, and $I = SR_+$ the ideal of S generated by R_+.

The following lemma is obtained by Steinberg [17].

Lemma 2.1. *Let $f \in S$ be a homogeneous polynomial. Then we have the following:*

(1) *$f \in I$ if and only if $f(\partial)\Delta = 0$,*

(2) *$g(\partial)f = 0$ for any $g \in I$ if and only if $f \in \mathbb{R}[\partial]\Delta$,*

(3) *I is the orthogonal complement of $\mathbb{R}[\partial]\Delta$, and $S = I \oplus \mathbb{R}[\partial]\Delta$.*

It is known that a W-stable graded subspace U of S such that $S = I \oplus U$ is isomorphic to the regular representation (see Bourbaki [1, Chap. 5 Sect. 5 Theorem 2]). By Lemma 2.1, the graded vector space $\mathbb{R}[\partial]\Delta$ affords the regular representation of W.

Assume that W is irreducible and V is generated by the roots. Then any endomorphism of V is a multiplicative map with a constant in $\mathbb{R}$, and V is absolutely irreducible (see Bourbaki [1, Chap. 5, Sect. 2, Proposition 1] and [1, Chap. 5, Sect. 3, Proposition 5]). Therefore, by the general theory of group representations, we have that the multiplicity of V in the regular representation is $\dim V = n$. Note that there exist a W-module U' such that $U = V^{\dim V} \oplus U'$.

Let M be a finite dimensional W-module, and set $U_k := U \cap S_k$ for any nonnegative integer k. Let $\{v'_1, \dots, v'_r\}$ be a basis of M, and $\{x'_1, \dots, x'_r\}$ the basis of M^* dual to $\{v_1, \dots, v_r\}$. Define a linear map

$$\theta : (U_k \otimes_{\mathbb{R}} M^*)^W \longrightarrow \mathrm{Hom}_W(M, U_k) \tag{2.1}$$

by $\theta(\sum u \otimes x)(v) = \sum x(v)u$ for any $\sum u \otimes x \in (U_k \otimes_{\mathbb{R}} M^*)^W$ and $v \in M$, and define a linear map

$$\eta : \mathrm{Hom}_W(M, U_k) \longrightarrow (U_k \otimes_{\mathbb{R}} M^*)^W \tag{2.2}$$

by $\eta(\phi) = \sum_{j=1}^{r} \phi(v'_j) \otimes x'_j$ for $\phi \in \mathrm{Hom}_W(M, U_k)$. It immediately follows from the correspondence above that

$$(U_k \otimes_{\mathbb{R}} M^*)^W \simeq \mathrm{Hom}_W(M, U_k). \tag{2.3}$$

In section 3, we use the isomorphisms θ (2.1) and η (2.2).

Lemma 2.2. *The vector space* $(U \otimes_{\mathbb{R}} V^*)^W$ *has dimension* n*.*

Proof. Since V is absolutely irreducible, the proof goes similarly to [13, Lemma 6.45]. $\square$

Lemma 2.2 gives us a method for constructing a basis for $(U \otimes V^*)^W$ by the representation theory. If we completely determine homomorphisms in $\mathrm{Hom}_W(V, U_k)$, then we have an $\mathbb{R}$-basis for $(U \otimes_{\mathbb{R}} V^*)^W$.

To determine canonical systems of basic invariants, we characterize the systems. Let $f_1, \ldots, f_n$ be a canonical system of basic invariants (Flatto [4, 5] and Flatto and Wiener [6]). Since $\{f_1, \ldots, f_n\}$ is linearly independent, the vector space $\langle f_1, \ldots, f_n \rangle_{\mathbb{R}}$ has dimension n. The vector space $\langle f_1, \ldots, f_n \rangle_{\mathbb{R}}$ does not depend on a choice of a canonical system, since a canonical system is uniquely determined up to equivalence.

Lemma 2.3. *Let* $f_1, \ldots, f_n$ *be a canonical system of basic invariants. Define a linear map*

$$\rho : \langle f_1, \ldots, f_n \rangle_{\mathbb{R}} \longrightarrow (\mathbb{R}[\partial]\Delta \otimes V^*)^W \tag{2.4}$$

by $\rho(f_i) = \sum_{j=1}^{n} \partial_j f_i \otimes x_j$ *for* $i = 1, \ldots n$*. Then the map* ρ *is an isomorphism from the vector space* $\langle f_1, \ldots, f_n \rangle_{\mathbb{R}}$ *to the vector space* $(\mathbb{R}[\partial]\Delta \otimes V^*)^W$*:*

$$\langle f_1, \ldots, f_n \rangle_{\mathbb{R}} \simeq (\mathbb{R}[\partial]\Delta \otimes V^*)^W.$$

Proof. For $f \in S^W$, $\sum_{j=1}^{r} \partial_j f \otimes x_j$ is W-invariant.

For $i, j, k = 1, \ldots, n$, we have $f_k(\partial)\partial_j f_i = 0$. Then we have by Lemma 2.1 that $\partial_j f_i \in \mathbb{R}[\partial]\Delta$ for $i, j = 1, \ldots, n$. By Lemma 2.2, $\dim(\mathbb{R}[\partial]\Delta \otimes V^*)^W = n = \dim\langle f_1, \ldots, f_n \rangle_{\mathbb{R}}$. To prove that ρ is an isomorphism, we only need to prove that ρ is injective.

Let f be an $\mathbb{R}$-linearly combination of $f_1, \ldots, f_n$ with $\rho(f) = 0$. Since $\rho(f) = \sum_{j=1}^{n} \partial_j f \otimes x_j$, we have $\partial_j f = 0$ for all $j = 1, \ldots, n$. The degree of f_i is greater than 0 for $i = 1, \ldots, n$. This implies $f = 0$. Hence ρ is injective. $\square$

3 Determining canonical systems

In this section, we assume that W is an irreducible finite reflection group and V is generated by the root system. Then the representation V is absolutely irreducible. Let $v_1, \ldots, v_n$ be an orthonormal basis for V, $x_1, \ldots, x_n$ the basis for V^* dual to $v_1, \ldots, v_n$, and $\partial_1, \ldots, \partial_n$ the basis for V^{**} dual to $x_1, \ldots, x_n$.

Let $f_1, \ldots, f_n$ be a canonical system of basic invariants.

Proposition 3.1. *The W-module*

$$M_i := \langle \partial_1 f_i, \dots, \partial_n f_i \rangle_{\mathbb{R}} \subseteq \mathbb{R}[\partial]\Delta \tag{3.1}$$

is isomorphic to V as a W-module for $i = 1, \dots, n$.

Proof. Define a W-homomorphism

$$\sigma_i : V = \langle v_1, \dots, v_n \rangle_{\mathbb{R}} \longrightarrow M_i = \langle \partial_1 f_i, \dots, \partial_n f_i \rangle_{\mathbb{R}} \tag{3.2}$$

by $\sigma_i(v_j) = \partial_j f_i$ for $j = 1, \dots, n$. We only need to prove $\ker \sigma_i = \{0\}$.

Assume that $\sigma_i(v_j) = 0$ for $j = 1, \dots, n$. This implies that $\partial_j f_i = 0$ for all $j = 1, \dots, n$. Then we have that the polynomial f_i is a constant. This contradicts $\deg f_i > 0$. Therefore we have $\ker \sigma_i = \{0\}$. $\square$

Let $m_1 \leq m_2 \leq \cdots \leq m_n$ be the degrees of elements of a system of basic invariants (which do not depend on a choice of a system of basic invariants). If W is not of type D_n with even n ($n \geq 4$), then the degrees are distinct: $m_1 < m_2 < \cdots < m_n$. If W is of type D_n with even n ($n \geq 4$), then the degrees are the numbers $2, 4, \dots, n, n, \dots, 2n-2$ (see [7, sect. 3.7 Table 1]).

3.1 Types except D_n with even n

In this subsection, we assume that W is not of type D_n with even n ($n \geq 4$). Then $m_1 < m_2 < \cdots < m_n$. We may assume $\deg f_i = m_i$ for $i = 1, \dots, n$.

Put $(\mathbb{R}[\partial]\Delta)_k := \mathbb{R}[\partial]\Delta \cap S_k$, and $\langle f_1, \dots, f_n \rangle_k := \langle f_1, \dots, f_n \rangle_{\mathbb{R}} \cap S_k$ for any non-negative integer k. Since the graded vector space $\mathbb{R}[\partial]\Delta$ is isomorphic to the regular representation, the multiplicity of V in $\mathbb{R}[\partial]\Delta$ is $\dim V = n$. We can find n W-submodules which are isomorphic to the W-module V in homogeneous components of $\mathbb{R}[\partial]\Delta$. By the correspondence (2.3) and Lemma 2.3, we have

$$\mathrm{Hom}_W(V, (\mathbb{R}[\partial]\Delta)_k) \simeq \langle f_1, \dots, f_n \rangle_{k+1} = \begin{cases} \mathbb{R} f_i & \text{if } k = m_i - 1, \\ \{0\} & \text{otherwise.} \end{cases}$$

Hence the multiplicity of V in the W-module $(\mathbb{R}[\partial]\Delta)_k$ is equal to 1 if $k = m_i - 1$, and 0 if $k \neq m_i - 1$ ($i = 1, \dots, n$). By Lemma 2.1 and Proposition 3.1, we have $V \simeq M_i \subseteq (\mathbb{R}[\partial]\Delta)_{m_i - 1}$.

Define a W-homomorphism

$$\phi : S \longrightarrow \mathbb{R}[\partial]\Delta, \ \phi(f) := ((f, \Delta), \Delta) \qquad \text{for } f \in S \tag{3.3}$$

where $(\cdot, \cdot)$ is the bilinear map (1.2). Then $\ker \phi = I$ by Lemma 2.1, and hence the map ϕ induces an automorphism of $\mathbb{R}[\partial]\Delta$ by Lemma 2.1 again.

Lemma 3.2. *The map ϕ induces an automorphism of M_i.*

Proof. There exists a W-submodule M_i' such that $(\mathbb{R}[\partial]\Delta)_{m_i-1} = M_i \oplus M_i'$ by Mascke's theorem. The projection $\pi : (\mathbb{R}[\partial]\Delta)_{m_i-1} \longrightarrow M_i'$ is a W-homomorphism. The multiplicity of V in $(\mathbb{R}[\partial]\Delta)_{m_i-1}$ is 1, hence $\pi \circ \phi|_{M_i} = 0$ by Schur's lemma.

Let $u \in M_i$, and write $\phi(u) = x + y$ for some $x \in M_i$ and $y \in M_i'$. Since $y = \pi(x + y) = \pi \circ \phi(u) = 0$, we have $\phi(u) = x \in M_i$. Therefore the image $\phi(M_i)$ is included in M_i. Since the restriction $\phi|_{\mathbb{R}[\partial]\Delta}$ is injective, then we have $\phi(M_i) = M_i$. □

By Lemma 3.2, we may regard $\phi|_{M_i}$ as the automorphism $\phi|_{M_i} : M_i \longrightarrow M_i$. Hence, by [1, Chap. 5, Sect. 2, Proposition 1], there exists a scalar $c_i \in \mathbb{R}^\times$ such that $\phi|_{M_i} = c_i 1$. The argument above give us the following proposition.

Proposition 3.3. *Fix $i = 1, \dots, n$. Every nonzero element in M_i is an eigenvector of ϕ with a nonzero eigenvalue. The eigenvalue does not depend on a choice of the elements, and depends only on index i.*

Theorem 3.4. *Let $h_1, \dots, h_n$ be an arbitrary system of basic invariants with* $\deg h_i = m_i$. *Then there exists a scalar $\lambda_i \in \mathbb{R}^\times$ such that*

$$\phi(\partial_j h_i) = \lambda_i \partial_j f_i$$

for $j = 1, \dots, n$. Therefore we have

$$\phi(\langle \partial_1 h_i, \dots, \partial_n h_i \rangle_{\mathbb{R}}) = M_i.$$

Proof. Since $h_1, \dots, h_n$ and $f_1, \dots, f_n$ are systems of basic invariants, we may write

$$h_i = d_i f_i + P(f_1, \dots, f_{i-1}) \tag{3.4}$$

for $i = 1, \dots, n$, where $d_i \in \mathbb{R}^\times$ and $P(f_1, \dots, f_{i-1})$ is a polynomial in $f_1, \dots, f_{i-1}$. Note that each term of P is a product of at least two polynomials of $f_1, \dots, f_{i-1}$.

Applying ∂_j on the equality (3.4), we have

$$\partial_j h_i = d_i \partial_j f_i + \partial_j P.$$

Since ∂_j is a derivation, for each term T of $\partial_j P$, there exists f_k $(i = 1, \dots, i-1)$ and a polynomial F such that $T = F f_k$. Then the polynomial $\partial_j P$ is in the ideal I, and we have $\phi(\partial_j P) = 0$ by Lemma 2.1. Hence we obtain

$$\phi(\partial_j h_i) = d_i \phi(\partial_j f_i) = d_i c_i \partial_j f_i$$

by Proposition 3.3. We complete the assertion by setting $\lambda_i := d_i c_i$ □

Let $h_1, \dots, h_n$ be a system of basic invariants with $\deg h_i = m_i$. By Theorem 3.4, for $i = 1, \dots, n$, there exists a nonzero scalar $\lambda_i \in \mathbb{R}^\times$ such that

$$\phi(\partial_j h_i) = \lambda_i \partial_j f_i. \tag{3.5}$$

Recall the W-isomorphism (3.2)

$$\sigma_i : V \longrightarrow M_i, \ \sigma_i(v_j) = \partial_j f_i \qquad (j = 1, \dots, n)$$

for $i = 1, \dots, n$. Since the W-module M_i is a unique submodule of $(\mathbb{R}[\partial]\Delta)_{m_i-1}$ which is isomorphic to V, we have

$$\mathrm{Hom}_W(V, M_i) \simeq \mathrm{Hom}_W(V, (\mathbb{R}[\partial]\Delta)_{m_i-1}).$$

We also recall the isomorphisms η (2.2) and ρ (2.4):

$$\mathrm{Hom}_W(V, (\mathbb{R}[\partial]\Delta)_{m_i-1}) \xrightarrow{\eta} ((\mathbb{R}[\partial]\Delta)_{m_i-1} \otimes_{\mathbb{R}} V^*)^W \xrightarrow{\rho^{-1}} \mathbb{R}f_i.$$

Then we have the following correspondence:

$$\begin{array}{ccccc} \mathrm{Hom}_W(V, M_i) & \xrightarrow{\eta} & (M_i \otimes_{\mathbb{R}} V^*)^W & \xrightarrow{\rho^{-1}} & \mathbb{R}f_i, \\ \sigma_i & \longleftrightarrow & \sum_{j=1}^n \partial_j f_i \otimes x_j & \longleftrightarrow & f_i = \frac{1}{m_i} \sum_{j=1}^n x_j \partial_j f_i. \end{array}$$

Hence we obtain the following explicit formula of a canonical system of basic invariants by (3.5).

Theorem 3.5. *Let $h_1, \dots, h_n$ be a system of basic invariants with $\deg h_i = m_i$. Then the system*

$$\sum_{j=1}^n x_j \phi(\partial_j h_1), \dots, \sum_{j=1}^n x_j \phi(\partial_j h_n)$$

form a canonical system of basic invariants, where $\phi(\partial_j h_i) = (((\partial_j h_i), \Delta), \Delta)$ for $i = 1, \dots, n$ with the bilinear map $(\cdot, \cdot)$ (1.2).

3.2 Type D_n with even n $(n \geq 4)$

Let $n = 2\ell$ $(\ell \geq 2)$. In this subsection, let W be the irreducible finite reflection group of type D_n. We may write that

$$\Delta = \prod_{1 \leq i < j \leq n} \left(x_i^2 - x_j^2\right) = \sum_{\boldsymbol{a} \in \mathbb{N}^n} c_{\boldsymbol{a}} x^{\boldsymbol{a}} \tag{3.6}$$

for some coefficients $c_{\boldsymbol{a}} \in \mathbb{R}$. If a multi-index $\boldsymbol{a}$ has a component with odd number, then the coefficient $c_{\boldsymbol{a}}$ in (3.6) is equal to 0.

Iwasaki [9] constructed a canonical system of basic invariants which contains the monomial $f_{\ell+1} = x_1 \cdots x_n$. We choice a canonical system $f_1, \ldots, f_\ell, f_{\ell+1}, \ldots, f_n$ of basic invariants which contains the monomial $f_{\ell+1} = x_1 \cdots x_n$. We may assume that $\deg f_1 < \cdots < \deg f_\ell = \deg f_{\ell+1} < \cdots < \deg f_n$. Then we have $\deg f_\ell = \deg f_{\ell+1} = \ell$. Note that the W-modules

$$M_\ell = \langle \partial_1 f_\ell, \ldots, \partial_n f_\ell \rangle_{\mathbb{R}} \text{ and } M_{\ell+1} = \langle \partial_1 f_{\ell+1}, \ldots, \partial_n f_{\ell+1} \rangle_{\mathbb{R}}. \tag{3.7}$$

are included in the same homogeneous component $(\mathbb{R}[\partial]\Delta)_{\ell-1}$.

We denote $|\boldsymbol{a}| := a_1 + \cdots + a_n$ for a multi-index $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$. We write $\boldsymbol{a} > \boldsymbol{b}$ if $a_k > b_k$ for all $k = 1, \ldots, n$. Put $\mathbf{1} := (1, \ldots, 1) \in \mathbb{N}^n$. Let $\boldsymbol{e}_j \in \mathbb{N}^n$ be the j-th unit vector of $\mathbb{N}^n$.

Theorem 3.6. *The monomial $\partial_j f_{\ell+1}$ is an eigenvector of ϕ with a nonzero eigenvalue for $j = 1, \ldots, n$. The eigenvalue does not depend on a choice of index j. In particular, the map ϕ induces an automorphism of $M_{\ell+1}$.*

Proof. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be multi-indices with $|\boldsymbol{a}| = |\boldsymbol{b}| = \deg \Delta$. Assume that all components of the multi-indices $\boldsymbol{a}$ and $\boldsymbol{b}$ are even, and write

$$\boldsymbol{a} = (2a_1, \ldots, 2a_n) \quad \text{and} \quad \boldsymbol{b} = (2b_1, \ldots, 2b_n).$$

For $j = 1, \ldots, n$, we obtain a sequence of equivalent conditions: $((\partial_j f_{\ell+1}, x^{\boldsymbol{a}}), x^{\boldsymbol{b}}) \neq 0$; if and only if $\boldsymbol{a} + \boldsymbol{e}_j > \mathbf{0}$ and $2b_k - 2a_k + 1 \geq 0$ for $k \neq j$ and $2b_j - 2a_j \geq 0$; if and only if $\boldsymbol{a} + \boldsymbol{e}_j > \mathbf{0}$ and $b_k \geq a_k$ for $k = 1, \ldots, n$. It follows from $|\boldsymbol{a}| = |\boldsymbol{b}|$ that $((\partial_j f_{\ell+1}, x^{\boldsymbol{a}}), x^{\boldsymbol{b}}) \neq 0$ if and only if $\boldsymbol{a} + \boldsymbol{e}_j > \mathbf{0}$ and $\boldsymbol{a} = \boldsymbol{b}$. Hence we conclude that

$$((\partial_j f_{\ell+1}, x^{\boldsymbol{a}}), x^{\boldsymbol{b}}) = \begin{cases} \lambda \partial_j f_{\ell+1} & \text{if } \boldsymbol{a} + \boldsymbol{e}_j > \mathbf{0},\ \boldsymbol{a} = \boldsymbol{b}, \\ 0 & \text{otherwise,} \end{cases}$$

for some $\lambda \in \mathbb{R}^\times$.

By (3.6), the polynomial $\phi(\partial_j f_{\ell+1}) = ((\partial_j f_{\ell+1}, \Delta), \Delta)$ is an $\mathbb{R}$-linearly combination of polynomials in

$$\{((\partial_j f_{\ell+1}, x^{\boldsymbol{a}}), x^{\boldsymbol{b}}) \mid \boldsymbol{a} = (2a_1, \dots, 2a_n)\,, \boldsymbol{b} = (2b_1, \dots, 2b_n) \in \mathbb{N}^n\}.$$

If $\boldsymbol{a} \neq \boldsymbol{b}$, then the polynomial $((\partial_j f_{\ell+1}, x^{\boldsymbol{a}}), x^{\boldsymbol{b}})$ is equal to 0. Hence there exists a scalar $c \in \mathbb{R}$ such that

$$\phi(\partial_j f_{\ell+1}) = c\partial_j f_{\ell+1} \in M_{\ell+1}.$$

This implies $\phi(M_{\ell+1}) \subseteq M_{\ell+1}$.

The scalar c does not depend on a choice of j by [1, Chap. 5, Sect. 2, Proposition 1]. Since $\phi|_{M_\ell} : M_{\ell+1} \longrightarrow M_{\ell+1}$ is injective, then we have $c \neq 0$. □

To verify that the polynomials $\partial_j f_\ell$ $(j = 1, \dots, n)$ are eigenvector of ϕ, we introduce three lemmas.

Lemma 3.7. *There exist scalars $d_1, d_2 \in \mathbb{R}$ such that*

$$\phi(\partial_j f_\ell) = d_1 \partial_j f_\ell + d_2 \partial_j f_{\ell+1}$$

for $j = 1, \dots, n$. The scalars d_1, d_2 do not depend on a choice of j.

Proof. There exists a W-stable vector space M' such that the W-module $\mathbb{R}[\partial]\Delta_{\ell-1}$ is decomposed into the direct sum of W-modules M_ℓ, $M_{\ell+1}$ and M':

$$\mathbb{R}[\partial]\Delta_{\ell-1} = M_\ell \oplus M_{\ell+1} \oplus M'. \tag{3.8}$$

Let π_3 be the projection from $\mathbb{R}[\partial]\Delta_{\ell-1}$ to M'. Since the multiplicity of V in $\mathbb{R}[\partial]\Delta_{\ell-1}$ is two, the homomorphism $\pi_3 \circ \phi|_{M_\ell}$ is 0. Hence we have by the decomposition (3.8) that $\phi(M_\ell)$ is included in $M_\ell \oplus M_{\ell+1}$.

Let π_1 be the projection from $(\mathbb{R}[\partial]\Delta)_{\ell-1}$ to M_ℓ. The map $\pi_1 \circ \phi|_{M_\ell}$ is endomorphism of M_ℓ. Thus we have that there exists a scalar $d_1 \in \mathbb{R}$ such that $\pi_1 \circ \phi|_{M_\ell} = d_1 \cdot 1_{M_\ell}$.

Let us consider the projection π_2 from $\mathbb{R}[\partial]\Delta_{\ell-1}$ to $M_{\ell+1}$. Recall maps $\sigma_\ell : V \longrightarrow M_\ell$ and $\sigma_{\ell+1} : V \longrightarrow M_{\ell+1}$ defined by

$$\sigma_\ell(v_j) = \partial_j f_\ell, \qquad \sigma_{\ell+1}(v_j) = \partial_j f_{\ell+1}$$

for $j = 1, \dots, n$. The maps σ_ℓ and $\sigma_{\ell+1}$ are W-isomorphisms. Any endomorphism of V is a multiplicative map. Thus there exists a scalar $d_2 \in \mathbb{R}$ such that $\sigma_{\ell+1}^{-1} \circ \pi_2 \circ$

$\phi|_{M_\ell} \circ \sigma_\ell = (d_2 \cdot 1_V)$: the diagram

$$\begin{array}{ccc} M_\ell & \xrightarrow{\pi_2 \circ \phi} & M_{\ell+1} \\ {\scriptstyle \sigma_\ell^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle \sigma_{\ell+1}^{-1}} \\ V & \xrightarrow{d_2 \cdot 1} & V \end{array}$$

is commutative.

Since $\phi|_{M_\ell} = (\pi_1 + \pi_2) \circ \phi|_{M_\ell}$, we have $\phi(\partial_j f_\ell) = d_1 \partial_j f_\ell + d_2 \partial_j f_{\ell+1}$. $\square$

Lemma 3.8. *For $j = 1, \ldots, n$, we have $\langle \partial_j f_{\ell+1}, \partial_j f_\ell \rangle = 0$.*

Proof. We write

$$f_\ell = \sum_{|\boldsymbol{a}|=n} c_{\boldsymbol{a}} x^{\boldsymbol{a}} \tag{3.9}$$

for some $c_{\boldsymbol{a}} \in \mathbb{R}$, then the coefficient $c_{\mathbf{1}}$ in (3.9) is equal to 0 since $\langle f_{\ell+1}, f_\ell \rangle = 0$.

Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be a multi-index with $|\boldsymbol{a}| = n$. Assume $\boldsymbol{a} \neq \mathbf{1}$. If $a_j = 0$, then $\partial_j x^{\boldsymbol{a}} = 0$. If $\boldsymbol{a} \neq \mathbf{1}$ and $a_j \neq 0$, then there exist at least one component a_k of $\boldsymbol{a}$ such that $a_k = 0$. This leads to

$$\langle \partial_j f_{\ell+1}, \partial_j x^{\boldsymbol{a}} \rangle = \langle x^{\mathbf{1}-\boldsymbol{e}_j}, x^{\boldsymbol{a}-\boldsymbol{e}_j} \rangle = 0.$$

Hence we have $\langle \partial_j f_{\ell+1}, \partial_j f_\ell \rangle = 0$. $\square$

Lemma 3.9. *For $j = 1, \ldots, n$, we have $\langle \partial_j f_{\ell+1}, \phi(\partial_j f_\ell) \rangle = 0$.*

Proof. Since the inner product of $\partial_j f_{\ell+1}$ and a monomial in $\phi(\partial_j f_\ell)$ except $x^{\mathbf{1}-\boldsymbol{e}_j} = \partial_j f_{\ell+1}$ is equal to 0, we only need to verify that the coefficient of $x^{\mathbf{1}-\boldsymbol{e}_j}$ in $\phi(\partial_j f_\ell)$ is equal to 0.

Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be a multi-index with $|\boldsymbol{a}| = n$. Let $\boldsymbol{b}$ and $\boldsymbol{c}$ be multi-indices with $|\boldsymbol{b}| = |\boldsymbol{c}| = \deg \Delta$, and assume that all components of the multi-indices $\boldsymbol{b}$ and $\boldsymbol{c}$ are even number. Write

$$\boldsymbol{b} = (2b_1, \ldots, 2b_n), \qquad \boldsymbol{c} = (2c_1, \ldots, 2c_n).$$

If $((x^{\boldsymbol{a}-\boldsymbol{e}_j}, x^{\boldsymbol{b}}), x^{\boldsymbol{c}})$ is a nonzero constant multiple of the monomial $x^{\mathbf{1}-\boldsymbol{e}_j}$, then $\boldsymbol{c} - \boldsymbol{b} + \boldsymbol{a} = \mathbf{1}$. Since all components of $\boldsymbol{b} - \boldsymbol{c}$ are even, then all components of $\boldsymbol{a}$ are odd number. This implies that $a_i \geq 1$ for $i = 1, \ldots, n$. Then we have $\boldsymbol{a} = \mathbf{1}$ since $|\boldsymbol{a}| = n$.

From the argument above, if $\boldsymbol{a} \neq \mathbf{1}$, then $((x^{\boldsymbol{a}-\boldsymbol{e}_j}, x^{\boldsymbol{b}}), x^{\boldsymbol{c}})$ is not a nonzero constant multiple of the monomial $x^{\mathbf{1}-\boldsymbol{e}_j}$. By $\langle f_{\ell+1}, f_\ell \rangle = 0$, the coefficient of $x_1 \cdots x_n$ in f_ℓ is 0. Hence the coefficient of $x^{\mathbf{1}-\boldsymbol{e}_j}$ in $\phi(\partial_j f_\ell)$ is equal to 0. $\square$

We have another proof of Lemma 3.9, but we have to prepare a technical tool for the proof. Hence we leave the proof behind. See Section 4.

Theorem 3.10. *The polynomial $\partial_j f_\ell$ is an eigenvector of ϕ with a nonzero eigenvalue for $j = 1, \dots, n$. The eigenvalue does not depend on a choice of index j. In particular, the map ϕ induces an automorphism of M_ℓ*

Proof. By Lemma 3.7, there exist scalars $d_1, d_2 \in \mathbb{R}$ such that $\phi(\partial_j f_\ell) = d_1 \partial_j f_\ell + d_2 \partial_j f_{\ell+1}$. Then it follows from Lemma 3.8 and Lemma 3.9 that

$$0 = \langle \partial_j f_{\ell+1}, \phi(\partial_j f_\ell)\rangle = d_1 \langle \partial_j f_{\ell+1}, \partial_j f_\ell\rangle + d_2 \langle \partial_j f_{\ell+1}, \partial_j f_{\ell+1}\rangle = d_2 \langle \partial_j f_{\ell+1}, \partial_j f_{\ell+1}\rangle$$

We have $d_2 = 0$ since $\langle \partial_j f_{\ell+1}, \partial_j f_{\ell+1}\rangle \neq 0$. Therefore we have $\phi(\partial_j f_\ell) = d_1 \partial_j f_\ell$. The map $\phi|_{M_{\ell+1}}$ is injective, then $d_1 \neq 0$. □

Let $h_1, \dots, h_\ell, h_{\ell+1}, \dots, h_n$ be a system of basic invariants. We may assume that $\deg h_i = m_i$ for $i = 1, \dots, n$. First observation, it follows from the argument in subsection 3.1 that $\phi(\partial_j h_i)$ is a nonzero constant multiple of $\partial_j f_i$ when i is neither ℓ nor $\ell + 1$.

Next, we consider h_ℓ and $h_{\ell+1}$. Then there exist polynomials $P_\ell, P_{\ell+1}$ in $f_1, \dots, f_{\ell-1}$ such that

$$\begin{aligned} h_\ell &= a_1 f_\ell + a_2 f_{\ell+1} + P_\ell(f_1, \dots, f_{\ell-1}), \\ h_{\ell+1} &= a_3 f_\ell + a_4 f_{\ell+1} + P_{\ell+1}(f_1, \dots, f_{\ell-1}), \end{aligned}$$

where

$$\begin{aligned} a_1 =& \frac{\langle f_\ell, h_\ell\rangle}{\langle f_\ell, f_\ell\rangle}, \qquad a_2 = \frac{\langle f_{\ell+1}, h_\ell\rangle}{\langle f_{\ell+1}, f_{\ell+1}\rangle}, \\ a_3 =& \frac{\langle f_\ell, h_{\ell+1}\rangle}{\langle f_\ell, f_\ell\rangle}, \quad a_4 = \frac{\langle f_{\ell+1}, h_{\ell+1}\rangle}{\langle f_{\ell+1}, f_{\ell+1}\rangle}. \end{aligned}$$

We have $a_1 a_4 - a_2 a_3 = \det \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \neq 0$ since $h_1, \dots, h_n$ and $f_1, \dots, f_n$ are systems of basic invariants. Therefore, by Theorem 3.6 and Theorem 3.10, the polynomial $\phi(\partial_j(a_4 h_\ell - a_2 h_{\ell+1}))$ (resp. $\phi(\partial_j(a_3 h_\ell - a_1 h_{\ell+1})))$ is a nonzero constant multiple of $\partial_j f_\ell$ (resp. $\partial_j f_{\ell+1}$). Hence we have the following form the similarly argument in subsection 3.1.

Theorem 3.11. *Let W be the irreducible finite reflection group of type D_n with $n = 2\ell$ ($\ell \geq 2$). Let $h_1, \dots, h_\ell, h_{\ell+1}, \dots, h_n$ be a system of basic invariants with* $\deg h_i = m_i$. *Put $f_{\ell+1} = x_1 \cdots x_n$. Then the system*

$$\sum_{j=1}^{n} x_j \phi(\partial_j h_1), \dots, \sum_{j=1}^{n} x_j \phi\left(\partial_j (a_4 h_\ell - a_2 h_{\ell+1})\right), f_{\ell+1}, \dots, \sum_{j=1}^{n} x_j \partial_j \phi(\partial_j h_n)$$

is a canonical system of basic invariants, where

$$a_2 = \frac{\langle f_{\ell+1}, h_\ell \rangle}{\langle f_{\ell+1}, f_{\ell+1} \rangle}, \qquad a_4 = \frac{\langle f_{\ell+1}, h_{\ell+1} \rangle}{\langle f_{\ell+1}, f_{\ell+1} \rangle}.$$

4 Appendix

In this section, we give another proof of Lemma 3.9. To give the proof, we introduce a technical tool (Proposition 4.1).

Let $h \in S$ be a homogeneous polynomial. We define a linear map $\phi_h : S \longrightarrow S$ by

$$\phi_h(f) := ((f, h), h)$$

for $f \in S$, where $(\cdot, \cdot)$ is the bilinear map (1.2). Note that $\phi_h(f) \in S_d$ for any $f \in S_d$. That is, the map ϕ_h preserves the degree of a homogeneous polynomial.

Proposition 4.1. *Let $f, g \in S$ be polynomials and $h \in S$ a homogeneous polynomial. Then we have*

$$\langle \phi_h(f), g \rangle = \langle f, \phi_h(g) \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product (1.1). That is, the map ϕ_h is symmetric with respect to the inner product $\langle \cdot, \cdot \rangle$.

Proof. Because of the bilinearity of the inner product $\langle \cdot, \cdot \rangle$, we can assume that f, g are monomials. If $\deg f \neq \deg g$ then $\langle \phi_h(f), g \rangle = 0 = \langle f, \phi_h(g) \rangle$ since ϕ_h preserves the degree of a homogeneous polynomial. Therefore we only need to verify that

$$\langle \phi_h(x^{\boldsymbol{a}}), x^{\boldsymbol{b}} \rangle = \langle x^{\boldsymbol{a}}, \phi_h(x^{\boldsymbol{b}}) \rangle \quad \text{for} \quad |\boldsymbol{a}| = |\boldsymbol{b}|. \tag{4.1}$$

We may write $h = \sum_{\boldsymbol{c} \geq 0} \lambda_{\boldsymbol{c}} x^{c}$ for some $\lambda_{c} \in \mathbb{R}$. Then we have

$$\begin{aligned}
\phi_h(x^{\boldsymbol{a}}) &= \left(\left(x^{\boldsymbol{a}}, \sum_{\boldsymbol{c} \geq \boldsymbol{0}} \lambda_{\boldsymbol{c}} x^{c} \right), \sum_{\boldsymbol{c}' \geq \boldsymbol{0}} \lambda_{\boldsymbol{c}'} x^{c'} \right) \\
&= \left(\sum_{\boldsymbol{c} \geq \boldsymbol{a}} \lambda_{c} \frac{\boldsymbol{c}!}{(\boldsymbol{c}-\boldsymbol{a})!} x^{\boldsymbol{c}-\boldsymbol{a}}, \sum_{\boldsymbol{c}' \geq \boldsymbol{0}} \lambda_{\boldsymbol{c}'} x^{c'} \right) \\
&= \sum_{\boldsymbol{c} \geq \boldsymbol{a}} \sum_{\boldsymbol{c}' \geq \boldsymbol{c}-\boldsymbol{a}} \lambda_{c} \lambda_{\boldsymbol{c}'} \frac{\boldsymbol{c}!}{(\boldsymbol{c}-\boldsymbol{a})!} \frac{\boldsymbol{c}'!}{(\boldsymbol{c}'-\boldsymbol{c}+\boldsymbol{a})!} x^{\boldsymbol{c}'-\boldsymbol{c}+\boldsymbol{a}},
\end{aligned}$$

where $\boldsymbol{a}! = a_1! \cdots a_n!$ for any multi-index $\boldsymbol{a} \in \mathbb{N}^n$. By the assumption $|\boldsymbol{a}| = |\boldsymbol{b}|$,

$$\begin{aligned}
\langle \phi_h(x^{\boldsymbol{a}}), x^{\boldsymbol{b}} \rangle &= \sum_{\boldsymbol{c} \geq \boldsymbol{a}} \sum_{\boldsymbol{c}' \geq \boldsymbol{c}-\boldsymbol{a}} \sum_{\boldsymbol{c}'-\boldsymbol{c}+\boldsymbol{a}=\boldsymbol{b}} \lambda_{c} \lambda_{\boldsymbol{c}'} \frac{\boldsymbol{c}!}{(\boldsymbol{c}-\boldsymbol{a})!} \frac{\boldsymbol{c}'!}{(\boldsymbol{c}'-\boldsymbol{c}+\boldsymbol{a})!} \boldsymbol{b}! \\
&= \sum_{\boldsymbol{c}-\boldsymbol{a} \geq 0} \sum_{\boldsymbol{c}' \geq \boldsymbol{c}-\boldsymbol{a}} \sum_{\boldsymbol{c}-\boldsymbol{a}=\boldsymbol{c}'-\boldsymbol{b}} \lambda_{c} \lambda_{\boldsymbol{c}'} \frac{\boldsymbol{c}!\, \boldsymbol{c}'!}{(\boldsymbol{c}-\boldsymbol{a})!}. \qquad (4.2)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\phi_h(x^{\boldsymbol{b}}) &= \left(\left(x^{\boldsymbol{b}}, \sum_{\boldsymbol{c}' \geq \boldsymbol{0}} \lambda_{\boldsymbol{c}'} x^{c'} \right), \sum_{\boldsymbol{c} \geq \boldsymbol{0}} \lambda_{\boldsymbol{c}} x^{c} \right) \\
&= \sum_{\boldsymbol{c}' \geq \boldsymbol{b}} \sum_{\boldsymbol{c} \geq \boldsymbol{c}'-\boldsymbol{b}} \lambda_{c'} \lambda_{\boldsymbol{c}} \frac{\boldsymbol{c}'!}{(\boldsymbol{c}'-\boldsymbol{b})!} \frac{\boldsymbol{c}!}{(\boldsymbol{c}-\boldsymbol{c}'+\boldsymbol{b})!} x^{\boldsymbol{c}-\boldsymbol{c}'+\boldsymbol{b}}.
\end{aligned}$$

By the assumption $|\boldsymbol{a}| = |\boldsymbol{b}|$ again,

$$\begin{aligned}
\langle x^{\boldsymbol{a}}, \phi_h(x^{\boldsymbol{b}}) \rangle &= \sum_{\boldsymbol{c}' \geq \boldsymbol{b}} \sum_{\boldsymbol{c} \geq \boldsymbol{c}'-\boldsymbol{b}} \sum_{\boldsymbol{a}=\boldsymbol{c}-\boldsymbol{c}'+\boldsymbol{b}} \lambda_{c'} \lambda_{\boldsymbol{c}} \frac{\boldsymbol{c}'!\, \boldsymbol{c}!}{(\boldsymbol{c}'-\boldsymbol{b})!} \\
&= \sum_{\boldsymbol{c}'-\boldsymbol{b} \geq 0} \sum_{\boldsymbol{c} \geq \boldsymbol{c}'-\boldsymbol{b}} \sum_{\boldsymbol{c}-\boldsymbol{a}=\boldsymbol{c}'-\boldsymbol{b}} \lambda_{c'} \lambda_{\boldsymbol{c}} \frac{\boldsymbol{c}'!\, \boldsymbol{c}!}{(\boldsymbol{c}-\boldsymbol{a})!}. \qquad (4.3)
\end{aligned}$$

If $\boldsymbol{c}-\boldsymbol{a} = \boldsymbol{c}'-\boldsymbol{b}$ then $\boldsymbol{c}' = \boldsymbol{c}-\boldsymbol{a}+\boldsymbol{b} \geq \boldsymbol{c}-\boldsymbol{a}$ and $\boldsymbol{c} = \boldsymbol{c}'-\boldsymbol{b}+\boldsymbol{a} \geq \boldsymbol{c}'-\boldsymbol{b}$. Moreover $\boldsymbol{c}-\boldsymbol{a} \geq 0$ is equivalent to $\boldsymbol{c}'-\boldsymbol{b} \geq 0$. This implies that the right hand side of (4.2) is equal to the right hand side of (4.3). Therefore (4.1) has been proved. $\square$

Thanks to Proposition 4.1, we can give a proof of Lemma 3.9.

Proof of Lemma 3.9. By Theorem 3.10, Lemma 3.8, and Proposition 4.1, we have

$$\langle \partial_j f_{\ell+1}, \phi(\partial_j f_\ell) \rangle = \langle \phi(\partial_j f_{\ell+1}), \partial_j f_\ell \rangle = \langle c_j \partial_j f_{\ell+1}, \partial_j f_\ell \rangle = 0$$

for $j = 1, \ldots, n$. $\square$

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